

The Chain Rule - Function within a function. Rules

* You must be very familiar with function decomposition

Suppose you are given $y = f(g(x))$

You must be able to identify $f(x)$ and $g(x)$

Its derivative is $\frac{dy}{dx} = f'(g(x))g'(x)$

Given $y = f(g(x))$

let $u = g(x)$, we have $y = f(u)$
and $\frac{du}{dx} = g'(x)$

Also, $f'(g(x)) = f'(u) = \frac{dy}{du}$

Therefore, another form of the chain rule is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 2p85 $y = -3(4x^3 + 8x - 1)^{-2}$

let $u = 4x^3 + 8x - 1$

$$\Rightarrow \frac{du}{dx} = 12x^2 + 8 \quad \text{AND} \quad y = -3u^{-2}$$

$$\Rightarrow \frac{dy}{du} = 6u^{-3}$$

$$\text{Thus, } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (6u^{-3})(12x^2 + 8)$$

$$\Rightarrow \frac{dy}{dx} = 6(4x^3 + 8x - 1)^{-3}(12x^2 + 8)$$

$$\text{OR } y = -3(4x^3 + 8x - 1)^{-2}$$

$$\Rightarrow \frac{dy}{dx} = 6(4x^3 + 8x - 1)^{-3}(12x^2 + 8)$$

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EXAMPLE 19P80

Find the equation of the tangent line to $f(x) = \sqrt[4]{x+1}$ at $x=15$.

You need two things:

↳ slope (m)↳ any one point on the line (x, y)

$$y - y_1 = m(x - x_1) \quad ; \text{ slope function is } f'(x)$$

$$\text{Let } u = x + 1$$

$$\Rightarrow \frac{du}{dx} = 1 + 0 = 1 \quad \text{ALSO, } f = \sqrt[4]{u}^{1/4}$$

$$\Rightarrow \frac{df}{du} = \frac{1}{4} u^{-3/4}$$

$$\text{Thus, } f'(x) = \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{1}{4} u^{-3/4} \right) (1) = \frac{1}{4} (x+1)^{-3/4}$$

∴ The slope of the tangent line at $x=15$ is

$$m = f'(15) = \frac{1}{4} (15+1)^{-3/4} = \frac{1}{4} (16)^{-3/4}$$

$$= \left(\frac{1}{4} \right) \left(\frac{1}{16} \right)^{3/4}$$

$$= \left(\frac{1}{4} \right) \left(\frac{1}{\sqrt[4]{16^3}} \right) x^{-n} = \frac{1}{x^{-n}}$$

$$= \left(\frac{1}{4} \right) \left(\frac{1}{2^3} \right) =$$

$$= \left(\frac{1}{4} \right) \left(\frac{1}{8} \right)$$

$$= \frac{1}{32}$$

We can use the point where the tangent line meets the curve as the point on the line that we need i.e. (X_1, Y_1)

So when $X=15$, we have $Y_1 = \sqrt[4]{15+1} = \sqrt[4]{16} = 2$

So, $(X_1, Y_1) = (15, 2)$ $\therefore$ the equation of the tangent line $Y - Y_1 = m(X - X_1)$

$$\Rightarrow Y - 2 = \frac{1}{32}(X - 15)$$

$$= \frac{1}{32}X - \frac{15}{32}$$

$$\Rightarrow Y = \frac{1}{32}X - \frac{15}{32} + 2$$

$$\Rightarrow Y = \frac{1}{32}X + \frac{49}{32}$$

Revenue Product - the relationship between revenue and the number of workers, $R(w)$, where the number of workers determines the quantity produced $q(w)$. In essence, the revenue is $R(q(w))$ which is the revenue product.

$$\Rightarrow \text{MARGINAL revenue product} = \frac{dR}{dw} = \frac{dR}{dq} \cdot \frac{dq}{dw}$$

which is the rate of change of the revenue at the w^{th} worker

Example

26P81

Given $R = 5q^2 + 225q$ and $q = 0.3w + 3.2w^{1/2}$, $0 \leq w \leq 10$

i) $R(w) = 5(0.3w + 3.2w^{1/2}) + 225(0.3w + 3.2w^{1/2})$

ii) $\frac{dR}{dw} = \frac{dR}{dq} \cdot \frac{dq}{dw}$ Given $R = 5q^2 + 225q$

$$\Rightarrow \frac{dR}{dq} = 10q + 225$$

and $q = 0.3w + 3.2w^{1/2}$

$$\Rightarrow \frac{dq}{dw} = 0.3 + 1.6w^{-1/2}$$

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$$\therefore \frac{dR}{dw} = \frac{dR}{dq} \cdot \frac{dq}{dw}$$

$$\Rightarrow \frac{dR}{dw} = (10q + 225) \cdot (0.3 + 1.6w^{1/2})$$

ii) At $w=10$, we have

$$q(10) = 0.3(10) + 3.2(10)^{1/2}$$

$$= 3 + 3.2\sqrt{10}$$

$$\therefore \frac{dR}{dw}(10) = (10(3 + 3.2\sqrt{10}) + 225(0.3 + 1.6(10)^{-1/2}))$$

$$= (30 + 32\sqrt{10} + 225)(0.3 + 1.6/\sqrt{10})$$

$$= (225 + 32\sqrt{10})(0.3 + 1.6/\sqrt{10})$$

$$\approx \$287.08$$

The Product Rule and the Quotient Rule

3.4

Product Rule

given $y = f(x)g(x)$, $\Rightarrow \frac{dy}{dx} = g(x)f'(x) + f(x)g'(x)$

OR $y = uv$, where u and v are functions in x

$$\Rightarrow \frac{dy}{dx} = vu' + uv'$$

Quotient Rule

given $y = \frac{f(x)}{g(x)}$ $\Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

OR $y = \frac{u}{v}$, where u and v are functions in x

$$\Rightarrow \frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

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EXAMPLE 3 P85 (chain rule)

$$y = (x^4 + 7\sqrt{x} + 1) \cdot (5x^3 + 1)^{-3}$$

$$\Rightarrow \frac{dy}{dx} = v u' + u v'$$

$$\text{Let } u = x^4 + 7\sqrt{x} + 1 = x^4 + x^{1/2} + 1$$

$$\Rightarrow u' = 4x^3 + \frac{1}{2}x^{-1/2}, \text{ and } v = (5x^3 + 1)^{-3}$$

$$\Rightarrow v' = -3(5x^3 + 1)^{-4} (15x^2)$$

$$\therefore \frac{dy}{dx} = (5x^3 + 1)^{-3} (4x^3 + \frac{1}{2}x^{-1/2}) + (x^4 + 7\sqrt{x} + 1) (-3(5x^3 + 1)^{-4} (15x^2))$$

10p85 (quotient rule)

$$y = \frac{\sqrt{10-x}}{\sqrt[3]{x+1}} \leftarrow u$$

$$\sqrt[3]{x+1} \leftarrow v$$

$$\text{let } u = \sqrt{10-x} = (10-x)^{1/2}$$

$$\Rightarrow u' = \frac{1}{2}(10-x)^{-1/2}(-1)$$

$$\text{let } v = \sqrt[3]{x+1} = (x+1)^{1/3}$$

$$\Rightarrow v' = \frac{1}{3}(x+1)^{-2/3}(1)$$

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v u' - u v'}{v^2}$$

$$\frac{dy}{dx} = \frac{\sqrt[3]{x+1} \left(\frac{1}{2}(10-x)^{-1/2}(-1) \right) - \sqrt{10-x} \left(\frac{1}{3}(x+1)^{-2/3} \right)}{(3\sqrt[3]{x+1})^2}$$

$$= \frac{(x+1)^{1/3}(-1/2)(10-x)^{-1/2} - (10-x)^{1/2}(x+1)^{-2/3}}{(3\sqrt[3]{x+1})^2}$$

$$= -\frac{(x+1)^{-2/3}(10-x)^{-1/2} \left[\frac{1}{2}(x+1) + \frac{1}{3}(10-x) \right]}{(x+1)^{2/3}}$$

$$= -\frac{\left(\frac{1}{2}x + \frac{1}{2} + \frac{10}{3} - \frac{1}{3}x \right)}{(x+1)^{4/3}(10-x)^{1/2}}$$

$$= \left(\frac{1}{6}x + \frac{23}{6} \right)$$

$$\frac{(1/6x + 23/6)}{(x+1)^{4/3}(10-x)^{1/2}}$$

$$= \frac{-(x+23)}{(x+1)^{4/3}(10-x)^{1/2}}$$

$$\frac{-(x+23)}{(x+1)^{4/3}(10-x)^{1/2}}$$

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13P85 Find the equation of the tangent line to

$$f(x) = 6x\sqrt{x+3} \text{ at } x=1$$

Slope function - $f'(x)$ given $f(x) = \overset{u}{6x} \overset{v}{\sqrt{x+3}}$

$$\text{let } u = x, \quad u' = 1 \quad \text{and} \quad v = \sqrt{x+3} = (x+3)^{1/2} \Rightarrow v' = \frac{1}{2}(x+3)^{-1/2}(1)$$

$$\text{So, } f'(x) = 6 [vu' + uv']$$

$$= 6 [\sqrt{x+3}(1) + x(\frac{1}{2}(x+3)^{-1/2})]$$

When $x=1$, the slope of the tangent line at $x=1$ is

$$m = f'(1) = 6 [\sqrt{1+3} + (1)(\frac{1}{2}(1+3)^{-1/2})]$$

$$= 6 [\sqrt{4} + (\frac{1}{2})(\frac{1}{4})]$$

$$= 6 [2 + (\frac{1}{2})(\frac{1}{2})]$$

$$m = 6 [2 + \frac{1}{4}]$$

$$= 6 (\frac{9}{4})$$

$$= \frac{27}{2}$$

Let (x_1, y_1) be the point where the tangent line touches the curve, so when $x=1$, we have

So,

$$(x_1, y_1) = (1, 12)$$

$$\begin{aligned} y_1 &= 6(1)\sqrt{1+3} \\ &= (6)\sqrt{4} \\ &= 12 \end{aligned}$$

∴ Equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$= y - 12 = \frac{27}{2}(x - 1)$$

$$= \frac{27}{2}x - \frac{27}{2} + 12$$

$$= \frac{27}{2}x - \frac{3}{2}$$